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Electric Currents in Multiply Connected Spaces

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It is shown that the topology of 3-space determines whether or not a current can be maintained in a wire loop.

1. INTRODUCTION

It is commonplace that one can maintain a current in a wire loop C by using, say, replaceable batteries or a superconducting wire. It is rather surprising, then, that this is possible only because ordinary space R^3 has a very simple *topology*; every closed curve C in R^3 has 0 intersection number with each closed 2-sided surface. I investigate here the problems that occur in maintaining a current in a multiply connected space. Such spaces are allowed by general relativity (indeed, we may live in one), and appear in other aspects of physics, e.g., when one replaces R^3 by the 3-torus T^3 when considering periodic boundary conditions.

In Figure 1, the curve C has 0 intersection number with each closed surface but C^* does not.

I emphasize that I am considering solutions of Maxwell's equations in a given background space-time.

2. SOME DEFINITIONS

See Frankel (1979) and Misner *et al.* (1973) for the use of differential forms in electromagnetism. In the present work, however, I use the term "pseudo" as a replacement for "twisted" in Frankel. **d** denotes the *spatial* exterior differential.

Maxwell's equations are

$$\mathbf{d}\mathscr{E}^1 = -\partial\mathscr{B}^2/\partial t, \qquad \mathbf{d}\mathscr{B}^2 = 0$$

$$\mathbf{d}\mathscr{H}^1 = 4\pi j^2 + \partial\mathscr{D}^2/\partial t, \qquad \mathbf{d}\mathscr{D}^2 = 4\pi\sigma\, dx \wedge dy \wedge dz$$

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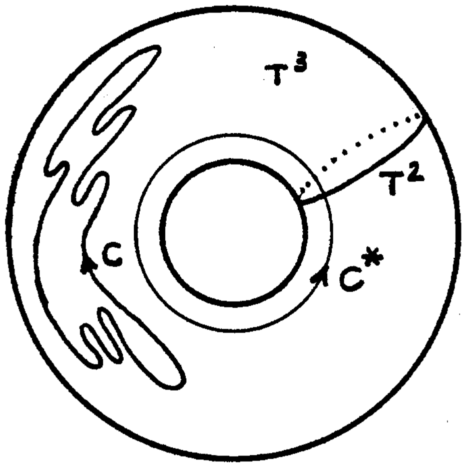

Fig. 1

I consider a space-time M^4 of the form $R \times V^3$ having a global time function t. The submanifolds $V(t_0)$ defined by $t = t_0$ are *the* spatial sections. The global vector field

$$\partial/\partial t := (\text{grad } t)/\|\text{grad } t\|^2$$

generates time translations $\phi_t: V(t_0) \mapsto V(t_0 + t)$.

Now let W^2 be a 2-sided surface (with given spatial normal N) sitting in the spatial section $t = 0$, and consider time translates $W_t = \phi_t(W)$. We say that a current through W is *maintained* for time T if the current 2-form j^2 satisfies

$$\left| \int_{W(t)} j^2 \right| = \left| \int_{W(t)} \langle \mathbf{J}, N \rangle \, \mathbf{d}S \right| \geq \varepsilon > 0$$

for some ε and for all $0 \leq t < T$. In the case of a current-carrying wire, this reduces to the usual concept if we let W be a cross section of the wire.

If V has local coordinates x^1, x^2, x^3, then these coordinates and $t = x^0$ can be used as coordinates in M. The ϕ_t keeps spatial coordinates constant. M has a metric of the form

$$ds^2 = g_{00}(t, \mathbf{x}) \, dt^2 + \sum_{\alpha,\beta=1}^{3} g_{\alpha\beta}(t, \mathbf{x}) \, dx^\alpha \, dx^\beta$$

We say that M is a *conformally static* universe if this metric can be written in the form

$$ds^2 = G^2(t, \mathbf{x})[-dt^2 + \sum h_{\alpha\beta}(\mathbf{x}) \, dx^\alpha \, dx^\beta]$$

where $h_{\alpha\beta}$ is a function only of x. This not only includes static spaces [where $G^2 = -g_{00}(x)$], but also Friedmann universes [where $G(t, x) = G(t)$ is a function of t alone and $h_{\alpha\beta}\, dx^\alpha\, dx^\beta$ is a metric of constant curvature].

3. SOME CONSEQUENCES OF MAXWELL'S EQUATIONS

Consider an M^4 with an electromagnetic field and a current 2-form j^2 (j is actually a *pseudo* form; a change of spatial orientation will send j into its negative).

Theorem 1. If W is a closed 2-sided surface whose area is bounded in time, then no current flux through W can be maintained for an infinite time.

Proof. Assume a current can be maintained. Choose the normal N to W so that

$$\int_{W(t)} j^2 \leq -\varepsilon < 0 \qquad \text{for} \quad t \geq 0$$

Let $\mathscr{H}^1$ be the magnetic pseudo 1-form and $\mathscr{D}^2$ the electric pseudo 2-form. From Ampere–Maxwell

$$\int_{W(t)} 4\pi j + \int_{W(t)} \partial\mathscr{D}/\partial t = \oint_{\partial W(t)} \mathscr{H} = 0$$

since W has no edge, $\partial W = 0$. Since W is spatially fixed,

$$d/dt \int_{W(t)} \mathscr{D} = \int_{W(t)} \partial\mathscr{D}/\partial t = -4\pi \int_{W(t)} j \geq 4\pi\varepsilon$$

for all $t > 0$. This would imply that

$$\int_{W(t)} \mathscr{D} \geq \text{const} + 4\pi\varepsilon t$$

Thus, if the current flux through W could be maintained for an infinite time, we would have

$$\int_{W(t)} \mathscr{D} \to \infty$$

Since W has bounded area, **E** would blow up on W, a physical impossibility. (As pointed out to me by W. Thompson, the increasing **E** field would make it increasingly difficult to maintain the current.) ■

Theorem 2. Let M be a conformally static universe (see definition above) and let C be a spatial wire loop that has 0 intersection number with every closed 2-sided surface in the spatial section V. Then a current in C can be maintained for the lifetime of the universe. (More precisely, there exists a solution of Maxwell's equations with $\mathbf{E}=0$, charge density zero, and where the current is constant and supported in the wire C.)

Proof. We shall need some lemmas.

Lemma 1. Let C be a closed curve in a 3-manifold V. There exists a closed pseudo 2-form j whose support lies in an ε-tube centered on C and whose flux through a cross section of this "wire" is positive. If C has intersection number 0 with each closed 2-sided surface in V, then there is a global (pseudo) 1-form $\mathscr{H}$ such that

$$d\mathscr{H}^1 = 4\pi j^2 \qquad \text{and} \qquad d^*\mathscr{H} = 0$$

Proof. Suppose first that V^3 is orientable. Then the normal bundle to C is trivial; one can find a smooth pair of orthonormal vector fields e_1, e_2 along C that are orthogonal to C. Then we can introduce "cylindrical" coordinates z, r, θ in some tubular neighborhood $r \le \varepsilon$ of C; z, r, θ is the unique point of V that is the endpoint of the geodesic segment of length r that leaves the curve C at parameter z, is orthogonal to C, and makes an angle θ with the first vector e_1. Now introduce a "bump function" $\rho(r)$ vanishing for $r \ge \varepsilon/2$ and put

$$j^2 = \rho(r)\, dr \wedge d\theta \tag{1}$$

This 2-form vanishes for $r \ge \varepsilon/2$ and can be extended to all of V by making it vanish outside of the wire. Certainly j is closed, $dj = 0$. We may also choose ρ so that

$$\int \rho(r)\, dr\, d\theta = 1 \tag{2}$$

When V is not orientable and the normal bundle to C is not trivial, we may still arrange to have e_1 return to itself while e_2 returns to its negative after going around C. In this case there is an ambiguity $\pm\theta$. But this simply means that the form j^2 defined in (1) is a *pseudo* form (as indeed it must be if it is to represent a current). We have proved the first part of Lemma 1.

If W^2 is a closed, 2-sided surface in V, we may always move W slightly so that this ε-tube strikes W "transversely." Then, from (2), we conclude that

$$\int_W j^2$$

is simply the intersection number of C with W, which vanishes by assumption. Thus, all the "periods" of the closed form j^2 vanish, and we conclude from de Rham's theorem that the form j is in fact exact,

$$4\pi j^2 = d\mathscr{H}^{\sim}$$

for some pseudo 1-form $\mathcal{H}^{\sim}$. By the usual process of adding the differential of a function to $\mathcal{H}^{\sim}$ and solving Poisson's equation, we may change $\mathcal{H}^{\sim}$ to a 1-form $\mathcal{H}^1$ that satisfies $d(*\mathcal{H}^1)=0$. [This is possible even in a closed V, since the integral of $d(*\mathcal{H}^{\sim})$ over V vanishes.] ■

Lemma 2. Consider the 3-manifold V of Lemma 1 as the spatial section of a static universe $R\times V^3$ with metric

$$ds^2=-dt^2+\sum_{\alpha,\beta=1}^{3} h_{\alpha\beta}(\mathbf{x})\,dx^\alpha\,dx^\beta \tag{3}$$

where h is the metric on V. Then if we consider the time-independent forms j^2 and $\mathcal{H}^1$ as being the current and magnetic field forms on $R\times V$, Maxwell's equations are satisfied on $R\times V$ if we put charge density and electric field $\mathbf{E}$ equal to 0.

Proof. $\mathcal{B}^2=\overset{*}{_{(4)}}(-\mathcal{H}^1\wedge dt)$, where the * operator is that for $R\times V^3$. Then $\partial\mathcal{B}/\partial t=0$ follows from the time independence of $\mathcal{H}^1$ and of $g_{00}=-1$, and $\mathbf{d}\mathcal{B}^2=0$ follows from $\mathbf{d}^*\mathcal{H}^1=0$ and $g_{00}=-1$. ■

Proof of Theorem 2. We have proven this in the case of a metric of the form (3). But a conformally static metric is conformal to (3) and Maxwell's equations are conformally invariant. ■

4. SOME CONCLUDING REMARKS AND QUESTIONS

The behavior of *current* fluxes contrasts strongly with that of *magnetic* fluxes. A nontrivial magnetic flux through a closed surface (which is possible in topologically complex spaces) is, in the absence of singularities, constant in time. This is an immedaite consequence of Faraday's law. This shows that *if the magnetic field can be made to vanish (say, by turning off the current in a circuit), then all the magnetic fluxes must have been zero and consequently this field must have admitted a global vector potential* **A**.

I wish to point out that while I have shown how to maintain a current in a conformally static universe, *I do not know if this result holds in a general space-time.* In a nonstatic space time one would expect an electric field to be generated, making it more difficult to satisfy Maxwell's equations for the entire lifetime of the universe.

I also should emphasize that Theorem 2 is simply a mathematical existence theorem; one shows the existence of a field (with $\mathbf{E}=0$) compatible with a mathematically constructed current. *I have not investigated how such a current and resulting field could be initiated!*

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